PHYSICAL REVIEW E, VOLUME 63, 045601(R)

Type of inversion problem in physics: An inverse emissivity problem

Tao Wen,[1] Denming Ming,[1] Xianxi Dai,[1,2,*] Jixin Dai,[3] and William E. Evenson[2]
[1]*Group of Quantum Statistics & Method of Theoretical Physics and Surface Physics Laboratory, Department of Physics, Fudan University, Shanghai 200433, People's Republic of China*
[2]*Department of Physics, Brigham Young University, Provo, Utah 84602-4645*
[3]*Department of Statistics, University of Wisconsin, Madison, Wisconsin 53706*

(Received 2 August 2000; published 19 March 2001)

Inversion problems have recently drawn vast amounts of attention from the physics community due to their potential widespread applications. In this Rapid Communication, a different type of inversion problem in physics is proposed: an inverse emissivity problem, which aims to determine the emissivity $g(\nu)$ by measuring only the total radiated power $J(T)$. Like other inverse problems, this one has potential for important practical applications. An exact solution is obtained for the proposed inverse problem. A unique existence theorem and techniques for eliminating divergences are also presented. A universal function set (UFS) suggested for numerical calculations is shown to be very useful in a numerical example. The UFS makes this inversion method practical and convenient for realistic calculations.

DOI: 10.1103/PhysRevE.63.045601 PACS number(s): 02.90.+p, 78.20.Ci

I. INTRODUCTION

Inversion problems are often important in physics and have received much interest and attention [1]. There are many familiar examples of inverse problems; for instance, inferring the velocity profile in the earth's crust from seismic signals. In the early 1980's, Bojarski first proposed a new inverse problem [2], namely, the black-body radiation inversion (BRI) problem. For a given or measured total power spectrum $W(\nu)$ radiated by a black-body with area-temperature distribution $a(T)$, the BRI problem is to solve the integral equation and obtain the area-temperature distribution function from the given $W(\nu)$. A series of important papers have been published and some imaginative methods have been proposed to solve the problem since then [2–5]. The cohesive energy-pair potential inversion problem was also proposed and developed in a way that anticipated the later work of Chen [6,7].

Specific heat-phonon spectrum inversion (SPI) is another type of interesting inversion problem. In many studies of high T_c superconductors, it is often of great importance to know the phonon spectrum. The SPI problem aims to obtain phonon spectra from specific-heat data, which in most cases are easier to obtain than direct measurements of phonon spectra. Much effort has been directed at this problem over the past decade [7,8]. Most recently, some of us with co-workers have been successful in numerical calculations of phonon spectra from the inversion of experimental specific heat data [9]. The numerical inversion results obtained from the exact solution [8] are in good agreement with the phonon spectrum from neutron inelastic scattering experiments [10].

In this Rapid Communication, we propose a different inversion problem: emissivity and transmissivity inversion (ETI). As is well-known, antiremote sensing is a very important and interesting problem in practice. It is of great importance for some flyers to hide their figures or images in the background from (infrared) detectors to protect themselves. One effective way to do this is to reduce their emissivity. Hence, the emissivity problem is potentially important and interesting.

In the case of gray-body radiation, if the emissivity $g(\nu)$ is known, then the total radiated power $J(T)$ can be written as

$$J(T)=\frac{2\,\pi\,h}{c^2}\int_0^\infty \frac{\nu^3\,g(\nu)\,d\,\nu}{\exp\left(\frac{h\nu}{k_B\,T}\right)-1}. \tag{1}$$

Here we propose the inverse emissivity problem: If the total radiated power $J(T)$ as a function of temperature can be measured, the emissivity $g(\nu)$ can be obtained by solving the integral equation, Eq. (1). Although $g(\nu)$ can be obtained by spectrum analysis, one usually needs sophisticated instruments that are suitable for many wave-bands. But in most cases, only the main or global characters of $g(\nu)$ are needed and only a single frequency-dependent detector is available. Therefore, an exact solution of the above integral equation is an important advance for a significant problem.

It should be emphasized here that our proposed inverse emissivity problem (ETI) is different from the previous black-body radiation inversion (BRI) and specific heat-phonon spectrum inversion (SPI) problems, since the integral kernels as well as the unknown functions are totally different. In addition, the physics of the three problems is not the same.

II. EXACT SOLUTION FOR THE ETI PROBLEM

In order to solve the equation exactly, one can use the transformation $x=\ln(T/T_0)$ and let $Q(x)=[J(T_0\,e^x)\,c^2h^3/2\,\pi\,(k_B\,T_0)^4]$. The following exact solution to Eq. (1) can then be obtained, using the Euler gamma function, $\Gamma(z)$, and the Riemann zeta function, $\zeta(z)$:

$$g(\nu)=\int_{-\infty}^\infty \left(\frac{h\,\nu}{k_B\,T}\right)^{ik-4}\frac{\tilde{Q}(k)\,dk}{\Gamma(ik)\,\zeta(ik)}, \tag{2}$$

*Corresponding author. Email address: xxdai@fudan.ac.cn

where $\tilde{Q}(k)$ is the Fourier transform of $Q(x)$.

But there are some difficulties in this formula: (i) $\zeta(ik)$ is not guaranteed to be nonzero in the denominator. Although, according to the Riemann hypothesis [11,12], all the zeros of Riemann zeta function $\zeta(z)$ in the Riemann strip ($0 \leqslant \mathrm{Re}\, z \leqslant 1$) are located on the line $\mathrm{Re}(z) = 1/2$, the Riemann hypothesis has never been proved and has been a famous unsolved problem in mathematics for over 100 years [11–15]; (ii) $\tilde{Q}(k)$ can be divergent in Eq. (2).

A. Exact solution formula and technique for eliminating divergence

We eliminate possible divergences by asymptotic behavior analysis of $Q(x)$. The fundamental requirement is the existence of $\tilde{Q}(k)$, so the first task is to control the asymptotic behavior of $Q(x)$. In general, $g(\nu) \leqslant 1$, so $J(T) \leqslant \sigma T^4$. Then assume that the measured $J(T)$ has the following asymptotic behavior:

$$J(T) \sim \begin{cases} T^{s_1} & \text{when } T \to \infty \\ T^{s_2} & \text{when } T \to 0. \end{cases} \tag{3}$$

By choosing

$$J(T)/T^s = \frac{2\pi h}{c^2} \int_0^\infty \frac{(\nu^3/T^s) g(\nu)\, d\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}, \tag{4}$$

with $s_1 < s < s_2$, we have: $\lim_{T\to 0} J(T)/T^s \to 0$; $\lim_{T\to\infty} J(T)/T^s \to 0$. Introducing the logarithmic transformation of the dependent variable T, the basic equation is transformed into

$$Q_0(x) = \int_{-\infty}^{\infty} K(y-x)\, F(y)\, dy, \tag{5}$$

where

$$y = \ln(h\nu/K_B T_0), \tag{6}$$

$$Q_0(x) = \frac{1}{2\pi} \frac{c^2 h^3}{(k_B T_0)^4} J(T_0 e^x) e^{-sx},$$

$$F(y) = e^{(4-s)y} g\left(\frac{k_B T_0}{h} e^y\right),$$

$$K(y-x) = \frac{e^{s(y-x)}}{\exp[e^{y-x}] - 1}.$$

One can prove that the Fourier transform of $K(x)$ is given by

$$\tilde{K}(-k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik\xi + s\xi}\, d\xi}{\exp(e^\xi) - 1} = \frac{1}{2\pi} \Gamma(s+ik)\, \zeta(s+ik). \tag{7}$$

And using a similar convolution theorem in Fourier transform, one has

$$\tilde{F}_0(k) = \frac{\tilde{Q}_0(k)}{\Gamma(s+ik)\, \zeta(s+ik)}, \tag{8}$$

and the exact solution formula of the emissivity inversion problem can be expressed by an inverse Fourier transform:

$$g(\nu) = \int_{-\infty}^{\infty} \frac{\tilde{Q}_0(k) \left(\frac{h\nu}{k_B T_0}\right)^{s+ik-4} dk}{\Gamma(s+ik)\, \zeta(s+ik)}. \tag{9}$$

B. Physical domain of *s*

The next important step is to give the conditions which guarantee the existence of $\tilde{F}_0(k)$. A clue is found by studying the general asymptotic behavior of $J(T)$ from physical considerations, followed by finding suitable constraints on the parameter s which is used for eliminating divergences. One can prove the following proposition.

Proposition: In physics, the largest domain of definition of s is $1 < s < \infty$.

Proof: The main point of the proof is to find lower and upper bounds of s_1 and s_2 for all possible $g(\nu)$ by physical analysis of the asymptotic behaviors of $J(T)$. This problem is rather difficult due to the absence of unified rules. But the following considerations lead in the right direction: (a) The integral equation is linear, so there exists a superposition principle; (b) $g(\nu)$ is positive definite and less than or equal to 1, i.e., $0 \leqslant g(\nu) \leqslant 1$.

A general source $g(\nu)$ can be considered as a superposition of point sources (i.e., $g(\nu)$ is concentrated at a single frequency). For a point source $g(\nu)$ of frequency ν_0, the total power is $J(T) \sim (2\pi h/c^2)[\nu_0^3/\exp(h\nu_0/k_B T) - 1]$. Its asymptotic behavior is

$$J(T) \sim \begin{cases} \frac{2\pi h}{c^2} \nu_0^3 \exp\left(-\frac{h\nu_0}{k_B T}\right) & \text{when } T \to 0 \\ \frac{2\pi}{c^2} \nu_0^2 k_B T \sim T & \text{when } T \to \infty. \end{cases}$$

In this limiting case, we have $s_1 = 1$, $s_2 = \infty$, and $1 < s < \infty$. Another limiting case is that of ideal black-body radiation: $g(\nu) \equiv 1$. Then $J(T) = \sigma T^4$. Considering $0 \leqslant g(\nu) \leqslant 1$ and comparing these two limiting cases, we conclude that $1 \leqslant s_1 \leqslant 4$, $4 \leqslant s_2 < \infty$, and the largest domain of definition of s is $1 < s < \infty$.

C. Exact solution formula and a unique existence theorem

If $\tilde{Q}_0(k)$ is continuous or has discontinuity points of the first kind, is monotonic at large k, and satisfies the following asymptotic behavior:

$$\tilde{Q}_0(k) = o[k^{s-(1/2)}\, e^{-k \tan^{-1} k/s}] \tag{10}$$

when $k \to \pm\infty$ and $1 < s$. Then the solution of the emissivity inversion equation exists uniquely, and the solution formula, Eq. (9), is exact.

Proof: In physics, the emissivity $g(\nu)$ must satisfy the condition $0 \leqslant g(\nu) \leqslant 1$, and the inversion equations are linear. In order to eliminate possible divergences of $\tilde{Q}'(k)$, one needs to introduce the parameter s. For all s in the largest domain of definition, one can guarantee $\zeta(s+ik) \neq 0$, i.e., the denominator in the exact solution is nonzero. This condition eliminates the divergence naturally and avoids the unproven Riemann hypothesis.

Another important condition is the asymptotic behavior of $\tilde{Q}_0(k)$. Condition (10) is necessary and sufficient to guarantee that the solution exists and is unique, by well-known Fourier transform uniqueness and existence theorems (e.g., see Ref. [17]).

The nature of our exact solution formula is different from the exact solution of other types of inversion problems. Here s cannot be chosen as $s>0$. On the contrary, when $0 \leqslant s \leqslant 1$, it just falls into the Riemann strip, and zeros could appear in the denominator. The superiority of this theorem is to give constraints on the asymptotic behavior of $J(T)$ by considering physical conditions. In the high temperature region of the SPI problem, $C_V(T) \rightarrow$constant due to the Dulong-Petit Law, and in the low temperature region $C_V(T) \rightarrow T^D$, where D is the dimensionality of the system. Then we have $0<s<3$. In SPI, this condition naturally avoids the Riemann hypothesis and cancels the divergence. In ETI, however, $1<s<\infty$, and at minimum $s>1$ is required. This condition naturally guarantees the denominator to be nonzero, cancels possible divergences of $\tilde{Q}_0(k)$, and simultaneously avoids the Riemann hypothesis via Hadamard's proof [16]. Therefore, based on the previous discussion, one can conclude that the natural laws are implicitly included in our solutions to these various inverse problems.

In summary,

$$1<s, \quad ETI$$

$$s=0, \quad BRI$$

$$0<s<3, \quad SPI.$$

In a limiting case, when $J(T) \sim T^4$, assume one has a radiation spectrum, proportional to T^4. Then

$$J(T) = \alpha_0 \sigma T^4, \tag{11}$$

and

$$Q_0(x) = \frac{c^2 h^3}{2\pi (k_B T_0)^4} J(T e^x) e^{sx} = c^2 h^3 \alpha_0 \sigma/(2\pi k_B^4). \tag{12}$$

Evidently, the only choice of s is

$$s = s_1 = s_2 = 4. \tag{13}$$

In order to include the limiting case, condition (10) can be relaxed to be $\tilde{Q}_0(k) = o[k^{s-(1/2)} \exp(-k \tan)^{-1}(k/s)]$. Although $\tilde{Q}'(k)$ cannot be expressed by a classical function, it can be expressed by a generalized function (or distribution) in the Schwartz-Sobolev sense:

$$\tilde{Q}_0(k) = \frac{c^2 h^3}{2\pi k_B^4} \alpha_0 \sigma \, \delta(-k). \tag{14}$$

According to the exact solution formula (9), with $s=4$, we have

$$g(\nu) = \frac{c^2 h^3 \alpha_0 \sigma}{2\pi k_B^4} \int_{-\infty}^{\infty} \frac{\left(\frac{h\nu}{k_B T_0}\right)^{s-4+ik} \delta(-k) dk}{\Gamma(s+ik)\,\zeta(s+ik)}$$

$$= \frac{c^2 h^3}{2\pi k_B^4} \frac{\alpha_0 \sigma}{\Gamma(4)\,\zeta(4)} = \alpha_0, \tag{15}$$

which means that the exact solution formula is still valid for the limiting case $s=4$.

It is necessary to emphasize that even in the black-body radiation case, if one does not introduce a parameter s, the Fourier transform cannot be used. The technique for eliminating divergences is necessary and important. The introduction of parameter s is also helpful to improve the asymptotic behavior of $Q_0(x)$ and reduce the amount of detail required in the temperature data.

III. UNIVERSAL FUNCTION SET (UFS) METHOD

According to the unique existence theorem, the special functions $\Gamma(z)$ and $\zeta(z)$ must be calculated to high precision. Because $\Gamma(s+ik)$ in the denominator goes to zero exponentially at large k, it is difficult to control the asymptotic behavior of $\tilde{Q}_0(k)$ at large k, which goes to zero much faster than $\Gamma(s+ik)$. The essence of our suggested UFS method is to choose a complete orthogonal function set to guarantee this asymptotic behavior in advance. We suggest choosing the Hermitian function set as the required basis:

$$u_n(x) = \sqrt{\left(\frac{\alpha}{\sqrt{\pi}\, 2^n n!}\right)} e^{-(1/2)\alpha^2 x^2} H_n(\alpha x), \tag{16}$$

where α is a parameter. One can expand $Q'(x)$ in terms of $u_n(x)$:

$$Q_0(x) = \sum_{n=0}^{\infty} C_n u_n(x). \tag{17}$$

Then the emissivity can be obtained as follows:

$$g(\nu) = \sum_{n=0}^{\infty} C_n G_n(\nu), \tag{18}$$

where

$$G_n(\nu) = \int_{-\infty}^{\infty} \frac{\tilde{u}_n(k) \left(\frac{h\nu}{K_B T_0}\right)^{s+ik-4} dk}{\Gamma(s+ik)\,\zeta(s+ik)}, \tag{19}$$

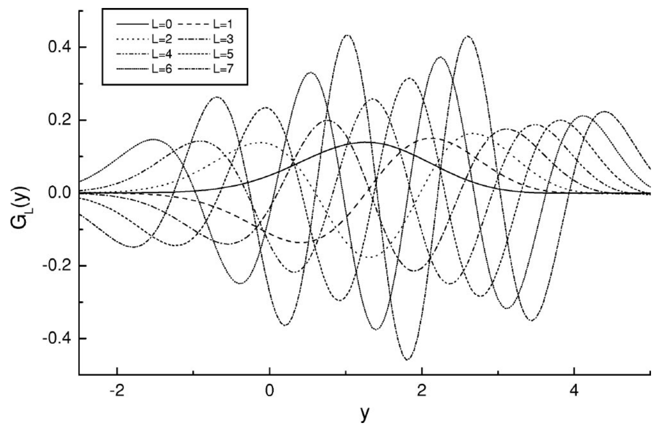

FIG. 1. Universal function set $G_l(y)$: $l=0-7$.

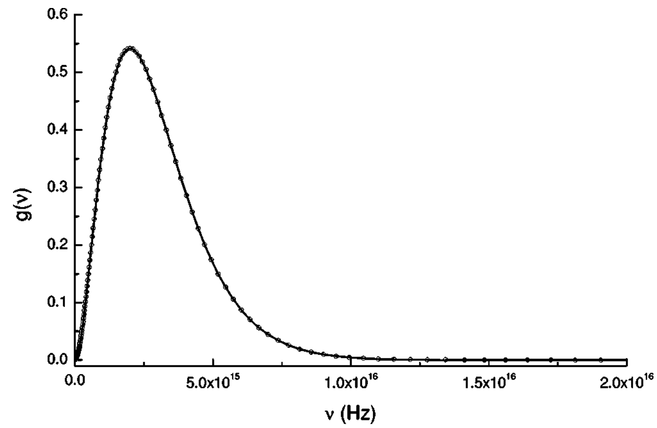

FIG. 2. Comparison of $g(\nu)$ calculated by the UFS method (open circles) with the known input function (solid curve).

and

$$\tilde{u}_n(k)=(-i)^n\sqrt{\frac{1}{2\,\pi\,\alpha\,\sqrt{\pi\,2^n\,n\,!}}}\,e^{-(k^2/2\,\alpha^2)}\,H_n\!\left(\frac{k}{\alpha}\right). \tag{20}$$

The universal function set $\{G_n(\nu)\}$ has been calculated to high precision. Some of these functions $[G_l(y)$: $l=0-7$, and y scaled as a dimensionless variable$]$ are shown in Fig. 1.

In order to check exact solution formula (9), we choose a known function $g(\nu)$ and obtain the corresponding $J(T)$ for input to ETI, as shown in the Fig. 2 by the solid curve. Then we calculate $g(\nu)$ by our UFS method, which is shown by the sample points in Fig. 2. Comparing the results obtained from the exact solution formula and from UFS with the input known function $g(\nu)$, we find excellent agreement. The addition of real noise introduces a number of new problems, as discussed by Bertero and Pike [18] in connection with noisy Laplace transforms.

IV. CONCLUSION

In this paper a new emissivity inversion problem is proposed that is expected to be useful in antiremote sensing and related fields. An exact solution formula with unique existence theorem, and a technique for eliminating divergences and avoiding the Riemann hypothesis are presented and proved. The largest physical definition domain of the parameter s proposed for eliminating divergences is found. The generalization of the ETI problem and some applications will be discussed in a subsequent publication.

ACKNOWLEDGMENTS

This work is in Project Nos. 19834010 and 19575009 and supported by the Chinese National Science Foundation.

[1] J. Maddox, Nature (London) **344**, 377 (1990).
[2] N. N. Bojarski, IEEE Trans. Antennas Propag. **30**, 778 (1982).
[3] Xianxi Dai and Jiqiong Dai, Phys. Lett. A **161**, 45 (1991), and references therein.
[4] Xianxi Dai and Jiqiong Dai, IEEE. Trans. Antennas Propag. **40**, 257 (1992).
[5] A. Lakhtakia, Phys. Lett. B **5**, 491 (1991), and references therein.
[6] A. E. Carlsson, C. D. Gelatt, and H. Ehrenreich, Philos. Mag. A **41**, 241 (1980).
[7] N. X. Chen, Phys. Rev. Lett. **64**, 1193 (1990); **64**, 3202(E) (1990); Q. Xie and N. X. Chen, Phys. Rev. E **52**, 6055 (1995).
[8] Xianxi Dai, Xinwen Xu, and Jiqiong Dai, Phys. Lett. A **147**, 445 (1990).
[9] Xianxi Dai, T. Wen, G. C. Ma, and J. X. Dai, Phys. Lett. A **264**, 68 (1999).
[10] B. Renker, F. Gompf, E. Gering, D. Ewert, H. Rietschel, and A. Dianoux, Z. Phys. B: Condens. Matter **73**, 309 (1988).
[11] G. Riemann, Monatsber. Berliner Akad. 671 (1859); Werke, 145.
[12] E. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed. revised (Oxford University Press, Oxford, 1986).
[13] A. Erdelyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vols. I, II, and III; see Vol. II, p. 119 3(a).
[14] E. T. Whittaker and G. N. Watson, *Modern Analysis*, 4th ed. (Cambridge University Press, Cambridge, England, 1934).
[15] Zhuxi Wang and Guo Dungren, *Introduction to Theory of Specific Functions* (Scientific, Beijing, 1979) (in Chinese).
[16] J. Hadamard, Bull. Soc. Math. France **24**, 199 (1896).
[17] G. M. Fihtengol'tz, *Course of Differential and Integral Calculus* (GITTL Press, Moscow, 1949), Vols. I–III; see Vol. III.
[18] M. Bertero and E. R. Pike, Inverse Probl. **7**, 21 (1991).